Vector Identity (6)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Proof

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) A_j B_k\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} A_j B_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \left(\frac{\partial A_j}{\partial x_i} B_k + A_j \frac{\partial B_k}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{ikj}) A_j \frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{lk} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_l - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{lj} \varepsilon_{ikj} A_l \frac{\partial B_k}{\partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_k) \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_l - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_j) \varepsilon_{ikj} A_l \frac{\partial B_k}{\partial x_i} \\
&= \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l B_l\right) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i}\right) - \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l A_l\right) \cdot \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \varepsilon_{ikj} \frac{\partial B_k}{\partial x_i}\right)
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l B_l \right) \cdot \left[\sum_{i=1}^{3} \sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial A_j}{\partial x_i} \right] - \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l A_l \right) \cdot \left[\sum_{i=1}^{3} \sum_{k=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_k) \frac{\partial B_k}{\partial x_i} \right] \\
&= \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l B_l \right) \cdot \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j \right) \right] - \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l A_l \right) \cdot \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k \right) \right] \\
&= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})
\end{aligned}$$